478. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Solve the equations, $l/x + y/m + z/n = 1$, $x/l + m/y + z/n = 1$, $x/l + y/m + n/z = 1$.

SOLUTION BY HENRY D. THOMPSON, Princeton N. J.

Take X, Y, Z as symbols for x/l, y/m, z/n, respectively. Then the equations to be solved are:

$$(1)\ 1/X + Y + Z = 1, \qquad (2)\ X + 1/Y + Z = 1, \qquad (3)\ X + Y + 1/Z = 1.$$

The difference of (2) and (1) gives (4) $(X - Y) + (X - Y)/XY = 0$, one solution of which is (5) $X = Y$. This substituted in (1), (2), (3) leaves to be solved (6) $1/X + X + Z = 1$, (7) $2X + 1/Z = 1$, and (7) gives $Z = 1/(1 - 2X)$.

This in (6) gives the cubic

$$1/X + 1/(1 - 2X) = 1 - X, \quad \text{or} \quad (1 - X)/X(1 - 2X) = (1 - X),$$

the solutions of which are $X = 1$, and $X = 1/4 \pm \sqrt{-7}/4$. These with (5) and (7) give for $[x, y, z]$ the three solutions:

$$\text{(I, II)}\ [(1 \pm \sqrt{-7})l/4,\ (1 \pm \sqrt{-7})m/4,\ (1 \pm \sqrt{-7})n/4], \qquad \text{(III)}\ [l, m, -n].$$

The other solution of (4) viz.: (8) $1 + 1/XY = 0$ in (1) or (2) gives $Z = 1$, whence (3) becomes $X + Y = 0$, which with (8) gives $X^2 - 1 = 0$, and $X = 1$ with $Y = -1$, or $X = -1$ with $Y = 1$. Two other solutions are:

$$\text{(IV)}\ [l, -m, n], \qquad \text{(V)}\ [-l, m, n].$$

To find the infinite solutions, set X/W, Y/W, Z/W, respectively, for X, Y, Z in (1), (2), (3) and these equations rid of denominators, and with $W = 0$ are:

$$(9)\ X(Y + Z) = 0, \qquad (10)\ Y(Z + X) = 0, \qquad (11)\ Z(X + Y) = 0.$$

The equation (9) is satisfied when $X = 0$, which set in (10) or (11) gives either $Y = 0$ or $Z = 0$. The other solution of (9) viz.: (12) $Y = -Z$ set in (10) and (11) gives $Y(X - Y) = 0$ and $-Y(X + Y) = 0$; whence $Y = 0$, and from (12), $Z = 0$. Thus the infinite solutions (VI), (VII), (VIII) are the points at infinity on the axes when x, y, z are taken as coördinates in a rectilineal system.

Note that each of the three hyperbolic cylinders (1), (2), (3) has for its asymptotic planes a coördinate plane and a plane parallel to the like named axis.

Also solved by B. J. DINING, GERTRUDE I. MCCAIN, J. L. RILEY, E. F. CANADY, S. W. REAVES, J. Q. MCNATT, G. W. HARTWELL, R. A. JOHNSON, C. H. WORTHINGTON, PAUL CAPRON, HORACE OLSON, and G. Y. SOSNOW.

479. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove

$$\left\{ \begin{vmatrix} x & -v & -z \\ -y & -z & v \\ -z & y & -x \end{vmatrix}^2 + \begin{vmatrix} y & -v & -z \\ x & -z & v \\ v & y & x \end{vmatrix}^2 + \begin{vmatrix} x & y & -z \\ -y & x & v \\ -z & v & -x \end{vmatrix}^2 + \begin{vmatrix} x & -v & y \\ -y & -z & x \\ -z & y & v \end{vmatrix}^2 \right\}$$

$$\div \begin{vmatrix} x & -y & -z & v \\ y & x & -v & -z \\ z & v & x & y \\ v & -z & y & -x \end{vmatrix}^2 = (x^2 + y^2 + z^2 + v^2)^{-1}.$$

SOLUTION BY HENRY D. THOMPSON, Princeton, N. J.

Let Q be a symbol for $(x^2 + y^2 + z^2 + v^2)$, and represent the determinants in the order in which they appear in the equation by Z, V, X, Y, H, so that the equation is

$$\{Z^2 + V^2 + X^2 + Y^2\} \div H^2 = Q^{-1}.$$

Application of the rule for the multiplication of determinants gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -yx & -yv \\ -xy & Q - x^2 & -xv \\ -vy & -vx & Q - v^2 \end{vmatrix}.$$

Dividing the first, second, and third *columns* by y, x, v, respectively, and *multiplying* the first, second, and third *rows*, respectively, by the same symbols gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -y^2 & -y^2 \\ -x^2 & Q - x^2 & -x^2 \\ -v^2 & -v^2 & Q - v^2 \end{vmatrix},$$

which easily reduces to $Z^2 = Q^2(Q - v^2 - y^2 - x^2) = Q^2z^2$.

In exactly the same way, $V^2 = Q^2v^2$, $X^2 = Q^2x^2$, $Y^2 = Q^2y^2$. Whence

$$\{Z^2 + V^2 + X^2 + Y^2\} = Q^2(z^2 + v^2 + x^2 + y^2) = Q^3.$$

Direct application of the rule for the multiplication of determinants gives for H^2 a determinant wherein each element in the leading diagonal is Q, and all the other elements are zero; whence $H^2 = Q^4$. These values set in the equation prove it to be correct.

Also solved by J. L. Riley, E. H. Worthington, A. M. Harding, J. B. Rosenbach, and the Proposer.

GEOMETRY.

511. Proposed by FRANK V. MORLEY, Student, Haverford, College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the incenter of the triangle formed by omitting a_i be c_i; prove that the four points c_i form a rectangle.

Solution by the Proposer.

Let there be four circles with centers, m_{12}, m_{23}, m_{34}, m_{41}, on the given circle, and let each circle intersect the next in two points, of which one set, a_2, a_3, a_4, a_1, lie on the given circle, and one set, c_4, c_1, c_2, c_3, lie inside, as shown in Fig. 1.

Then $\measuredangle c_4a_1a_2 = \frac{1}{2}\measuredangle c_4m_{12}a_2$, and $\measuredangle m_{23}a_1a_2 = \frac{1}{2}a_3m_{12}a_2$ (1).

$\therefore \measuredangle c_4c_3a_2 = \measuredangle c_4a_1a_2 = \measuredangle m_{23}a_1a_2 = \measuredangle m_{23}m_{41}a_2$, and $c_3c_4 \parallel m_{41}m_{23}$.

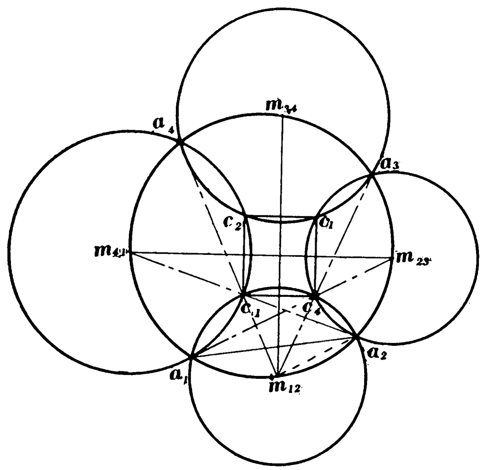

Fig. 1.

By equality of arcs, $m_{41}m_{23} \perp m_{12}m_{34}$, and hence $c_3c_4 \perp m_{12}m_{34}$.

Since $\measuredangle a_4m_{12}a_3 \equiv \measuredangle c_3m_{12}c_4$, and is bisected by $m_{12}m_{34}$, c_3, c_4 are reflections through $m_{12}m_{34}$, and likewise c_1, c_2; similarly c_1, c_4, and c_2, c_3, are reflections through $m_{41}m_{23}$.

$\therefore$ c_1, c_2, c_3, c_4, form a rectangle.